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The Quantum Poincaré group as Hidden Symmetry of General Relativity

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Abstract

Using the tools of q–differential calculus and quantum Lie algebras associated to quantum groups, we find a one–parameter family of q-gauge theories associated to the quantum group $ISO_q(3,1)$. Although the gauge fields, that is the spin–connection and the vierbeins are non–commuting objects depending on a deformation parameter, q, it is possible to construct out of them a metric theory which is insensitive to the deformation. The Christoffel symbols and the Riemann tensor are ordinary commuting objects. Hence it is argued that torsionless Einstein's General Relativity is the common invariant sector of a one–parameter family of deformed gauge theories.

Introduction

The description of general relativity as a gauge theory is a well established result in 2+1 dimensions [1], where in particular it turns out to be a Chern Simons theory associated to the Poincaré group $ISO(2,1)$. This equivalence allows for its extension as a q-gauge theory

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[2] once a deformed Chern–Simons theory is available. This result has been achieved in [3] where a deformed Chern–Simons action has been found for multi–parametric, minimally deformed, quantum groups. In [4] such a theory has been specialized to the case of the quantum Poincaré group $ISO_q(2,1)$ [5], yielding a description of 2+1 gravity as a q–gauge theory. The usual description in terms of dreibeins and spin–connections associated to the Poincaré group, $ISO(2,1)$, is replaced by deformed dreibeins and spin–connections associated to $ISO_q(2,1)$, which obey nontrivial braiding relations, the limit q=1 corresponding to the usual, undeformed theory. This yields a one–parameter family of Lagrangian and Hamiltonian formalisms for 2+1 gravity (the parameter being the deformation parameter). The metric tensor, which we need to recover Einstein's theory, is constructed as in the undeformed case as a suitable bilinear in the dreibeins. Then, the one–parameter family of Lagrangian descriptions (including the case q=1), has a common metric sector, that is to say, all the theories of the family are *equivalent* from the point of view of dynamics. To be more specific, though the underlying connection components are non–commuting objects and parameter dependent, the components of the metric tensor $g_{\mu\nu}$ commute among themselves and the field equations that they satisfy are formally identical to those of the ordinary theory. Remarkably, the relevant fields of the metric theory, such as Christoffel symbols and the Riemann tensor, turn out to be ordinary commuting fields having the standard form in term of the deformed metric. This situation is somewhat analogous to the case of classical fermionic field theories where commuting Dirac currents are constructed as bilinears in non–commuting fields.

Although the classical dynamics is equivalent to the usual one, that is not the case for the Hamiltonian formalism. The one parameter family of canonical formalisms associated to the deformed symmetry yields inequivalent theories. This feature should become significant when quantizing such theories. However, it must be mentioned that the problem of quantization for systems exhibiting q–symmetry is quite delicate and we do not know, at the moment, how to solve it.

In this contribution we will deal with the physically interesting 3+1 dimensional case. The $2+1$ case described above may be regarded as a toy model to be used as a guide. We will see that the final result, that is the existence of alternative descriptions for General Relativity relying on the quantum Poincaré group is preserved in the $3+1$ dimensional case [6]. We will find again a hidden quantum group structure in General Relativity, though this is achieved in a very non trivial manner. As it is well known, the major difference among the two theories is that 2+1 gravity may be described by a CS action, namely, as a gauge theory, while 3+1 gravity cannot be formulated as a topological theory. It can be described in terms of gauge potentials for the Poincaré group $ISO(3,1)$, but the action only exhibits an invariance under the local Lorentz subgroup. Only when we impose the torsion to be zero, is the action invariant with respect to the whole Poincaré group. It is torsionless General Relativity which we will deal with and we will show that there exists a whole one–parameter family of q–gauge theories associated to the q–Poincaré group $ISO_q(3,1)$, all having the same metric sector in common, exactly as for the 2+1 dimensional case. Again, the Christoffel symbols and the Riemann

tensor, constructed out of non–commuting connection components, are given by the usual expression in terms of the metric tensor, and they commute with *all* fields of the theory, hence being ordinary objects.

It should be stressed that the equivalence not only holds for the pure gravity case, but it also holds in the presence of matter, provided there are no sources for torsion. (In fact it is only a non–zero torsion which distinguishes the different classical theories from one another, each one coupling to a different kind of "exotic" matter.) We finally mention that a one–parameter family of Hamiltonian formalisms has been developed for the 3+1 dimensional case too. We will not in this paper be concerned with the canonical formalism. In the 2+1 dimensional case we refer the reader to [4] while for the 3+1 dimensional case a detailed analysis may be found in [6].

In section **1** we briefly review the well known formulation of General Relativity as a gauge theory. In section **2** we first describe the structure of q–deformed gauge theories mainly along the line of Refs. [2, 3], then we specialize to the q–Poincaré group and show that the gauge formulation of Einstein's General Relativity may be generalized to a one–parameter family of deformed gauge theories exhibiting local invariance with respect to the quantum Lie algebra associated to the quantum group $ISO_q(3,1)$. We conclude with brief final remarks.

1 General Relativity as a Gauge Theory

Before discussing the q-Poincaré theory it is useful to briefly review the description of general relativity as a Poincaré group gauge theory, so that the deformation procedure will be better understood. The Poincaré group $ISO(3,1)$ may be parameterized by a Lorentz matrix ℓ_{ab} and Lorentz vector z_a, $a, b = 1, ...4$. The former satisfies the constraints of the connected Lorentz group

$$\ell_{ab}\ell_c{}^b = \ell_{ba}\ell^b{}_c = \eta_{ac} \ , \quad \det \parallel \ell_{ab} \parallel = 1 \ , \tag{1.1}$$

where η_{ab} is the Lorentz metric. The associated Lie algebra is spanned by ten generators T_i, which we split into M_{ab} (the Lorentz generators) and P_a (the translation generators), satisfying

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \eta_{ac}M_{bd} - \eta_{bc}M_{ad} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc} \\
[M_{ab}, P_c] &= -(\eta_{bc}P_a - \eta_{ac}P_b) \\
[P_a, P_b] &= 0 \ .
\end{aligned} \tag{1.2}$$

On the space of *commutative* functions generated by ℓ_{ab} and z_a (the Lie group manifold), one can construct the usual differential geometry of Maurer-Cartan forms on the Poincaré group. The Maurer-Cartan form can be elevated to connection one form $A(x) = A^a_\mu T_a dx^\mu$, which may be split into spin connection $\omega(x) = \omega^{ab}_\mu(x) M_{ab} dx^\mu$ and vierbein one form $e(x) = e^a_\mu(x) P_a dx^\mu$. The derivative

$$\mathcal{D}_\mu = \partial_\mu + A_\mu(x) \equiv \partial_\mu + A^i_\mu(x) T_i \ , \tag{1.3}$$

is covariant with respect to infinitesimal gauge transformations of the form

$$\begin{aligned} \delta\omega_\mu^{ab} &= \partial_\mu \tau^{ab} + \omega_\mu^{ac}\ \tau_c{}^b - \omega_\mu^{bc}\ \tau_c{}^a\ , \\ \delta e_\mu^c &= \partial_\mu \rho^c + \omega_\mu^{cb}\ \rho_b - \tau^c{}_b\ e_\mu^b\ , \end{aligned} \tag{1.4}$$

where the gauge parameters $\tau^{ab} = -\tau^{ba}$ and ρ^a are associated with Lorentz transformations and translations, respectively.

The field strength tensor, F, given by

$$F = dA + A^2\ , \tag{1.5}$$

may be split into the Lorentz curvature $\mathcal{R} = \mathcal{R}_{\mu\nu}^{ab}(x) M_{ab} dx^\mu \wedge dx^\nu$ and the torsion $\mathcal{T} = \frac{1}{2}\mathcal{T}_{\mu\nu}^a(x) P_a dx^\mu \wedge dx^\nu$, where

$$\begin{aligned} \mathcal{R}_{\mu\nu}^{ab} &= \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} - [\omega_\mu, \omega_\nu]^{ab} \\ \mathcal{T}_{\mu\nu}^a &= \partial_\mu e_\nu^a - \partial_\nu e_\mu^a - [\omega_\mu, e_\nu]^a\ . \end{aligned} \tag{1.6}$$

The dynamics of the theory is determined by a locally Lorentz invariant action

$$S = \frac{1}{4}\int_M \epsilon_{abcd} \mathcal{R}^{ab} \wedge e^c \wedge e^d \tag{1.7}$$

which can be put into the Palatini form

$$S = \frac{1}{2}\int d^4x\ \mathtt{e}\ e_a^\mu e_b^\nu \mathcal{R}_{\mu\nu}^{ab}\ . \tag{1.8}$$

Here $\mathtt{e} \equiv \det e_\mu^a$ and e_a^μ denotes the inverse of the vierbein fields. By taking variations with respect to the vierbeins and spin–connections we obtain the equations of motion:

$$e_b^\nu \mathcal{R}_{\mu\nu}^{ab} - \frac{1}{2} e_\mu^a (e_c^\rho e_b^\nu \mathcal{R}_{\rho\nu}^{cb}) = 0\ , \qquad \mathcal{T}_{\mu\nu}^a = 0\ . \tag{1.9}$$

To recover Einstein's theory we have to re-express the Palatini action in terms of the space–time metric and scalar curvature. The space–time metric is introduced as a bilinear in the vierbeins

$$\mathtt{g}_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b\ , \tag{1.10}$$

which is symmetric and invariant with respect to local Lorentz transformations. The Christoffel symbols $\Gamma_{\mu\nu}^\sigma$ are then defined by demanding the covariant derivative of the vierbeins to vanish

$$0 = \mathtt{D}_\mu e_\nu^b = \partial_\mu e_\nu^b + \omega_\mu^{bc} e_{\nu c} - \Gamma_{\mu\nu}^\sigma e_\sigma^b\ . \tag{1.11}$$

By multiplying this expression by $\eta_{ab} e_\rho^a$, summing over the b index, and symmetrizing with respect to the space-time indices ν and ρ, we eliminate the spin–connection, getting

$$0 = \eta_{ab}[e_\rho^a \partial_\mu e_\nu^b + e_\nu^a \partial_\mu e_\rho^b - e_\rho^a e_\sigma^b \Gamma_{\mu\nu}^\sigma - e_\nu^a e_\sigma^b \Gamma_{\mu\rho}^\sigma]\ . \tag{1.12}$$

Adding to this the equation obtained by switching μ and ν, and subtracting the equation obtained by replacing indices (μ,ν,ρ) by (ρ,μ,ν), we finally obtain the Christoffel symbols in the form

$$2\eta_{ab}e^a_\rho e^b_\sigma \Gamma^\sigma_{\mu\nu} = \eta_{ab}[e^a_\rho(\partial_\mu e^b_\nu + \partial_\nu e^b_\mu) + e^a_\nu(\partial_\mu e^b_\rho - \partial_\rho e^b_\mu) + e^a_\mu(\partial_\nu e^b_\rho - \partial_\rho e^b_\nu)] \tag{1.13}$$

or

$$2\mathtt{g}_{\rho\sigma}\Gamma^\sigma_{\mu\nu} = \partial_\mu \mathtt{g}_{\rho\nu} + \partial_\nu \mathtt{g}_{\rho\mu} - \partial_\rho \mathtt{g}_{\nu\mu} \ . \tag{1.14}$$

The Riemann tensor is then defined as

$$\begin{aligned} \mathtt{R}_{\mu\nu\rho}{}^\sigma v_\sigma &= (\mathtt{D}_\mu \mathtt{D}_\nu - \mathtt{D}_\nu \mathtt{D}_\mu) v_\rho = \\ &- [\partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\sigma_{\mu\tau}\Gamma^\tau_{\nu\rho} - \Gamma^\sigma_{\nu\tau}\Gamma^\tau_{\mu\rho}] v_\sigma \ , \end{aligned} \tag{1.15}$$

where v_μ is an arbitrary covector. Its relation with the Lorentz curvature is then given by:

$$\mathtt{R}_{\mu\nu\rho}{}^\tau = -\mathcal{R}_{\mu\nu}{}^a{}_b e^b_\rho e^\tau_a \ . \tag{1.16}$$

Thus it can be checked that the action (1.8) takes the usual Einstein–Hilbert form

$$S = \frac{1}{2}\int d^4x \ \sqrt{-\mathtt{g}} \ \mathtt{R} \tag{1.17}$$

where $\mathtt{g} \equiv \det \mathtt{g}_{\mu\nu}$ and $\mathtt{R}$ is the scalar curvature $\mathtt{R} = \mathtt{R}_{\nu\mu\rho}{}^\mu \mathtt{g}^{\nu\rho}$. The equations of motion now read

$$\mathtt{R}_{\mu\nu} - \frac{1}{2}\mathtt{g}_{\mu\nu}\mathtt{R} = 0 \ , \quad \mathcal{T}^\rho_{\mu\nu} = 0 \tag{1.18}$$

where $\mathcal{T}^a_{\mu\nu} = e^a_\rho \mathcal{T}^\rho_{\mu\nu}$.

This discussion can be easily extended to take into account spinless matter. The matter action has to be added to (1.8), so that the right hand side of the first of equations (1.9) will be proportional to $\theta^a_\mu = \frac{\partial \mathcal{L}}{\partial e^\mu_a}$, which is strictly related to the energy–momentum tensor, while the second equation is unchanged. Then the right hand side of (1.18) is proportional to the energy momentum tensor $T_{\mu\nu} = \mathtt{g}_{\rho\nu} e^\rho_a \theta^a_\mu$.

2 General Relativity as a q–Gauge Theory

In this section we will show that the equivalence between Poincaré gauge theory and Einstein gravity can be extended to the case where the local invariance with respect to the Lie algebra of $ISO(3,1)$ is replaced by local invariance with respect to the *quantum Lie algebra* associated to the quantum Poincaré group $ISO_q(3,1)$.

Let us first recall the definition of a quantum Lie algebra and its connection to differential calculus on quantum groups, as described in [7]. Starting from the definition of a quantum group G_q as the non–commutative algebra of functions on the Lie group G, $G_q \equiv Fun_q(G)$, a bimodule of left (right) invariant forms for G_q is constructed, in the same way as the bimodule

of left (right) invariant forms is constructed for classical Lie groups. Such a bimodule inherits the non–commutative nature of the product in $Fun_q(G)$,

$$R^{ab}_{ef}\,N^e_c N^f_d = N^b_f N^a_e\,R^{ef}_{cd}\ , \tag{2.19}$$

(N is an element of G in its defining representation, while R is the R–matrix, satisfying the quantum Yang Baxter equation) so that the usual definition of exterior product for one–forms, $\theta^i \wedge \theta^j = \theta^i \otimes \theta^j - \theta^j \otimes \theta^i$, is replaced on q–groups by

$$\theta^i \wedge \theta^j = \theta^i \otimes \theta^j - \Lambda^{ij}_{kl}\,\theta^k \otimes \theta^l \tag{2.20}$$

where Λ is the braiding matrix. Following the analogy with the differential calculus on classical Lie groups, the algebra of left invariant vector fields, which is dual to the algebra of left invariant one-forms, can be obtained,* with q–commutation relations

$$T_i T_j - \Lambda^{kl}_{ij}\,T_k T_l \equiv [T_i, T_j]_q = C^k_{ij} T_k\ . \tag{2.21}$$

It is this algebra which is called the quantum Lie algebra. In the limit $q \to 1$, $\Lambda^{kl}_{ij} \to \delta^k_j \delta^l_i$ and T_i become the generators of the classical Lie algebra. C^k_{ij} are q-structure constants, which in general are not antisymmetric in the lower two indices except in the limit $q \to 1$. In order to define a bicovariant calculus, the braiding matrix Λ and the structure constants have to satisfy the following relations [2]:

$$\Lambda^{ij}_{kl}\Lambda^{lm}_{sp}\Lambda^{ks}_{qu} = \Lambda^{jm}_{kl}\Lambda^{ik}_{qs}\Lambda^{sl}_{up} \quad \text{(Yang Baxter equation)} \tag{2.22}$$

$$C^r_{mi}C^n_{rj} - \Lambda^{kl}_{ij}C^r_{mk}C^n_{rl} = C^k_{ij}C^n_{mk} \quad (q\text{-Jacobi}) \tag{2.23}$$

$$\Lambda^{ir}_{mk}\Lambda^{ks}_{nl}C^j_{rs} = \Lambda^{ij}_{kl}C^k_{mn}\ , \tag{2.24}$$

$$\Lambda^{jq}_{ri}\Lambda^{si}_{kl}C^r_{ps} + \Lambda^{jq}_{pi}C^i_{kl} = C^j_{is}\Lambda^{sq}_{rl}\Lambda^{ir}_{pk} + C^q_{rl}\Lambda^{jr}_{pk}\ . \tag{2.25}$$

The first condition is the quantum Yang Baxter equation; the second is the Jacobi identity for the algebra (2.21), while the last equations are trivial in the limit $q \to 1$.

Following [2], the gauge potential is assumed to be a q–Lie algebra valued one–form $A \equiv A^i_\mu T_i dx^\mu$. In this approach the deformation occurs solely in the fiber and thus the A^i_μ are taken to be q–fields subject to nontrivial commutation relations. Space–time, instead, remains an ordinary manifold so that dx^μ are ordinary space–time differentials commuting with A^i_μ. The exterior product of one–forms on the space–time manifold is deformed in the same way as the exterior product of invariant forms on the group manifold (2.20) and, for general groups, one has:

$$A^i \wedge A^j = -Z^{ij}_{kl}\,A^k \wedge A^l\ ; \tag{2.26}$$

where Z is a matrix of ordinary $c-$numbers which depends on the group. The undeformed case obviously corresponds to the choice $Z^{ij}_{kl} = \delta^i_l \delta^j_k$ for any group. Deformed Chern Simons theories

*In the same way we can introduce right invariant objects. Bicovariant differential calculus [8] requires that left and right actions of the q–group on the bimodule commute.

have been constructed only for minimal deformations [3], that is deformations satisfying $\Lambda^2 = 1$; moreover, only minimal deformations are known for the inhomogenous groups like $ISO(3,1)$ which we are interested in [9]. In this case the matrix Z has a simple expression in terms of the braiding matrix Λ:

$$A^i \wedge A^j = -\Lambda^{ij}_{kl} A^k \wedge A^l \ , \tag{2.27}$$

The braiding matrix Λ will depend in general on a set of parameters q_i and on r. The number of independent parameters depends on the group. For $ISO_q(3,1)$ there si only one parameter, which we will indicate as q. The deformed gauge transformations are assumed to be of the usual form

$$\delta_\epsilon A = -d\epsilon - A\epsilon + \epsilon A \tag{2.28}$$

where $\epsilon \equiv \epsilon^i T_i$. The gauge parameters ϵ^i are now q–numbers and are assumed to have the following commutation rules with the gauge fields:

$$\epsilon^i A^j = \Lambda^{ij}_{mn} A^m \epsilon^n \ . \tag{2.29}$$

The commutation relations for A^i with $d\epsilon^j$ and dA^i with ϵ^j can be obtained by taking the exterior derivative of the above equation and imposing that the terms containing dA^i and ϵ^j cancel separately. The field strength is defined in the usual way

$$F \equiv \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = dA + A^2 \ , \tag{2.30}$$

where $A^2 = A^i A^j T_i T_j$. F is valued in the deformed Lie-algebra [2] and under a gauge transformation (2.28) it transforms as:

$$\delta_\epsilon F = \epsilon F - F\epsilon \ . \tag{2.31}$$

Let us specialize now to the q–Poincaré group. The group manifold is parameterized by a Lorentz vector z_a and a Lorentz matrix ℓ_{ab} so that (2.19) is replaced by

$$z^a \ {\ell_c}^b = q^{\Delta(b)} \ {\ell_c}^b \ z^a \ , \tag{2.32}$$

where

$$\Delta(1) = -1 \ , \quad \Delta(2) = \Delta(3) = 0 \ , \quad \Delta(4) = 1 \ , \tag{2.33}$$

and all other commutation relations are trivial. The Lorentz metric tensor is taken to be the following off-diagonal matrix:

$$\eta = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix} \ . \tag{2.34}$$

Then the commutation relations (2.32) are consistent with the Lorentz constraints (1.1) due to the identity

$$\eta_{ab} = q^{\Delta(a)+\Delta(b)} \eta_{ab} \ , \tag{2.35}$$

$ISO_q(3,1)$ thus contains the undeformed Lorentz group. The braiding matrix, Λ, appearing in (2.21) and (2.27) is given by

$$\begin{aligned} \Lambda^{cd\ ab}_{ab\ cd} = 1 \ , \quad \Lambda^{b\ a}_{a\ b} = q^{\Delta(a)-\Delta(b)} \ , \\ \Lambda^{a\ bc}_{bc\ a} = (\Lambda^{bc\ a}_{a\ bc})^{-1} = q^{\Delta(b)+\Delta(c)} \ , \end{aligned} \tag{2.36}$$

with all other components vanishing. In terms of the Lorentz and translation generators, M_{ab} and P_a, the quantum Lie algebra (2.21) is expanded to

$$\begin{aligned} [M_{ab}, M_{cd}] = \eta_{ac} M_{bd} - & \ \eta_{bc} M_{ad} + \eta_{bd} M_{ac} - \eta_{ad} M_{bc} \\ [M_{ab}, P_c]_{q^{\Delta(a)+\Delta(b)}} & = -(\eta_{bc} P_a - \eta_{ac} P_b) \\ [P_a, P_b]_{q^{\Delta(a)-\Delta(b)}} & = 0 \ , \end{aligned} \tag{2.37}$$

where $[\alpha, \beta]_s \equiv \alpha\beta - s\beta\alpha$. Eqs. (2.37) reduce to (1.2) for $q = 1$. For each value of q the quantum Lie-algebra contains the undeformed Lorentz algebra.

Splitting the connection one–form into spin–connection, $\omega(x) = \omega^{ab}_\mu(x) M_{ab} dx^\mu$ and vierbein one–form, $e(x) = e^a_\mu(x) P_a dx^\mu$, as in the undeformed case, and assuming the space–time to be spanned by ordinary commutative coordinates, (2.27) implies

$$\begin{aligned} \omega^{ab}_\mu \omega^{cd}_\nu = \omega^{cd}_\nu \omega^{ab}_\mu \ , \\ e^a_\mu \omega^{bc}_\nu = q^{\Delta(b)+\Delta(c)} \ \omega^{bc}_\nu e^a_\mu \ , \\ e^a_\mu e^b_\nu = q^{\Delta(b)-\Delta(a)} \ e^b_\nu e^a_\mu \ . \end{aligned} \tag{2.38}$$

Splitting the gauge parameter ϵ^i into Lorentz and translation parameter, τ, ρ respectively, the gauge transformations (2.28) become

$$\begin{aligned} \delta\omega^{ab} &= d\tau^{ab} + \omega^a{}_c\, \tau^{cb} - \omega^b{}_c\, \tau^{ca} \ , \\ \delta e^c &= d\rho^c + \omega^c{}_b\, \rho^b - \tau^{cb}\, e_b \ . \end{aligned} \tag{2.39}$$

From (2.29) we get the following commutation relations between gauge parameters and one–forms

$$\begin{aligned} \rho^a\, \omega^{bc} &= q^{\Delta(b)+\Delta(c)}\, \omega^{bc}\, \rho^a \\ \rho^a\, e^b &= q^{\Delta(b)-\Delta(a)}\, e^b\, \rho^a \\ \tau^{ab}\, e^c &= q^{-\Delta(a)-\Delta(b)}\, e^c\, \tau^{ab} \\ \tau^{ab}\omega^{cd} &= \omega^{cd}\tau^{ab} \ . \end{aligned} \tag{2.40}$$

Finally the curvature and the torsion $\mathcal{R}^{ab}$ and $\mathcal{T}^a$, have the usual expressions

$$\begin{aligned} \mathcal{R}^{ab} &= d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \ , \\ \mathcal{T}^a &= de^a + \omega^a{}_b \wedge e^b \ , \end{aligned} \tag{2.41}$$

though they obey non trivial braiding relations with the connection components, which can be obtained by (2.38).

Next we write down a locally Lorentz invariant action:

$$\mathcal{S} = \frac{1}{4} \int_M \epsilon_{abcd} \mathcal{R}^{ab} \wedge \mathcal{E}^{cd} , \tag{2.42}$$

where $\mathcal{E}^{cd}$ is the two–form

$$\mathcal{E}^{cd} = -\mathcal{E}^{dc} = q^{-\Delta(d)} e^c \wedge e^d , \tag{2.43}$$

M is a four manifold and ϵ_{abcd} is the ordinary, totally antisymmetric tensor with $\epsilon_{1234} = 1$. The expression (2.42) differs from that of the undeformed case by the $q^{-\Delta(d)}$ factor. Note that this factor can be written differently using the identity

$$q^{\Delta(a)+\Delta(b)+\Delta(c)+\Delta(d)} \ \epsilon_{abcd} = \epsilon_{abcd} \ . \tag{2.44}$$

As in the undeformed case, the action is invariant under the full set of local Poincaré transformations (2.39), provided we impose the torsion to be zero upon making the variations.

The equations of motion obtained from varying the vierbeins have the usual form, i.e.

$$\epsilon_{abcd} \mathcal{R}^{ab} \wedge e^c = 0 , \tag{2.45}$$

while varying ω^{ab} gives

$$d\tilde{\mathcal{E}}_{ab} = \omega_a{}^c \tilde{\mathcal{E}}_{bc} - \omega_b{}^c \tilde{\mathcal{E}}_{ac} , \qquad \tilde{\mathcal{E}}_{ab} \equiv \epsilon_{abcd} \mathcal{E}^{cd} . \tag{2.46}$$

Due to the antisymmetry of $\mathcal{E}^{cd}$, we get the following expression in terms of the torsion form (2.46)

$$\epsilon_{abcd} \mathcal{T}^c \wedge e^d \ q^{-\Delta(d)} = 0 . \tag{2.47}$$

We will show next that this equation implies zero torsion, provided inverse vierbeins exist. This is necessary in order to recover Einstein's gravity.

3 Recovering Einstein's theory

We now prove that the metric formulation of the q-deformed Cartan theory of gravity discussed above is completely equivalent to the *undeformed* Einstein's theory, for all values of q.

To make a connection with Einstein gravity, we need to introduce the space-time metric $\mathsf{g}_{\mu\nu}$ on M. As in the undeformed case it has to be a bilinear in the vierbeins which is symmetric in the space-time indices and invariant under local Lorentz transformations. These requirements uniquely fix $\mathsf{g}_{\mu\nu}$ to be

$$\mathsf{g}_{\mu\nu} = q^{\Delta(a)} \ \eta_{ab} \ e^a_\mu e^b_\nu , \tag{3.48}$$

Using eqs.(2.38) we see that $\mathsf{g}_{\mu\nu}$ is symmetric, although the tensor elements are not c-numbers since

$$\mathsf{g}_{\mu\nu} \ \omega^{ab}_\rho = q^{2\Delta(a)+2\Delta(b)} \ \omega^{ab}_\rho \ \mathsf{g}_{\mu\nu} ,$$

$$\mathsf{g}_{\mu\nu}\ e^a_\rho = q^{2\Delta(a)}\ e^a_\rho\ \mathsf{g}_{\mu\nu}\ . \tag{3.49}$$

The components of $\mathsf{g}_{\mu\nu}$ do however commute with themselves.

The inverse e^μ_a of the vierbeins e^a_μ can be defined if we enlarge our algebra by a new element e^{-1} such that:

$$\mathsf{e}^{-1} e^a_\mu \;=\; q^{-4\Delta(a)} e^a_\mu\ \mathsf{e}^{-1}\ , \tag{3.50}$$

$$\mathsf{e}^{-1}\omega^{ab}_\mu \;=\; q^{-4(\Delta(a)+\Delta(b))}\omega^{ab}_\mu\ \mathsf{e}^{-1} \tag{3.51}$$

$$\mathsf{e}^{-1}\mathsf{e} \;=\; 1\ , \tag{3.52}$$

where e is the determinant:

$$\mathsf{e} = \epsilon^{\mu\nu\rho\sigma} e^1_\mu e^2_\nu e^3_\rho e^4_\sigma\ . \tag{3.53}$$

Eq.(3.52) is consistent because its left hand side commutes with everything, due to eqs.(3.51). Moreover, one can check that $\mathsf{e}^{-1}\mathsf{e} = \mathsf{e}\mathsf{e}^{-1}$. The inverse of the vierbeins can now be written:

$$e^\mu_a = \frac{1}{3!}\hat{\epsilon}_{abcd}\epsilon^{\mu\nu\rho\sigma} e^b_\nu e^c_\rho e^d_\sigma \mathsf{e}^{-1}\ , \tag{3.54}$$

where the totally q-antisymmetric tensor $\hat{\epsilon}_{abcd}$ is defined such that

$$\hat{\epsilon}_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \quad \text{no sum on } a,b,c,d \tag{3.55}$$

The solution to this equation can be expressed by

$$\hat{\epsilon}_{abcd} = q^{3\Delta(a)+2\Delta(b)+\Delta(c)+3}\ \epsilon_{abcd}\ . \tag{3.56}$$

Notice also the following useful identity satisfied by the q-antisymmetric tensor $\hat{\epsilon}^{abcd}$ obtained by raising the indices of $\hat{\epsilon}_{abcd}$ with the metric η^{ab}

$$q^{-6}\ \hat{\epsilon}^{abcd}\ \mathsf{e} = -\epsilon^{\mu\nu\lambda\sigma} e^a_\mu e^b_\nu e^c_\lambda e^d_\sigma\ . \tag{3.57}$$

The explicit expression of $\hat{\epsilon}^{abcd}$ can be seen to be:

$$\hat{\epsilon}^{abcd} = q^{-3\Delta(a)-2\Delta(b)-\Delta(c)+3}\ \epsilon^{abcd}\ , \tag{3.58}$$

where ϵ^{abcd} is the ordinary antisymmetric tensor obtained by raising the indices of ϵ_{abcd} with the metric η^{ab}. It is easy to prove that the inverse of the vierbeins (3.54) have the usual properties:

$$e^a_\mu e^\mu_b = e^\mu_b e^a_\mu = \delta^a_b\ ,$$

$$e^a_\mu e^\nu_a = e^\nu_a e^a_\mu = \delta^\nu_\mu\ . \tag{3.59}$$

By using the inverse of the vierbeins, we can now prove that eq.(2.47) implies the vanishing of the torsion. To begin with, we introduce the components of the torsion two-form along the vierbeins:

$$\mathcal{T}^a_{bc} \;\equiv\; q^{\Delta(b)}\mathcal{T}^a_{\mu\nu} e^\mu_b e^\nu_c\ ; \tag{3.60}$$

the power of q ensures that they are antisymmetric in the lower indices, $\mathcal{T}^a_{bc} = -\mathcal{T}^a_{cb}$. Now we rewrite eq.(2.47) as:

$$\begin{aligned} 0 &= q^{-\Delta(d)}\ \epsilon_{abcd}\epsilon^{\lambda\mu\nu\rho}\mathcal{T}^c_{\mu\nu}e^d_\rho = q^{-\Delta(d)-\Delta(h)}\ \epsilon_{abcd}\epsilon^{\lambda\mu\nu\rho}\mathcal{T}^c_{gh}e^g_\mu e^h_\nu e^d_\rho = \\ &= -q^{\Delta(d)-\Delta(f)-3}\epsilon_{abcd}\epsilon^{fghd}\mathcal{T}^c_{gh}e^\lambda_f \mathtt{e} = \\ &= 2q^{-\Delta(a)-\Delta(b)-\Delta(c)-3}(q^{-\Delta(a)}\mathcal{T}^c_{bc}e^\lambda_a + q^{-\Delta(b)}\mathcal{T}^c_{ca}e^\lambda_b + q^{-\Delta(c)}\mathcal{T}^c_{ab}e^\lambda_c)\ \mathtt{e}\ , \end{aligned} \tag{3.61}$$

where we have used the identity

$$q^{-\Delta(d)-\Delta(h)}\epsilon^{\lambda\mu\nu\rho}e^g_\mu e^h_\nu e^d_\rho = -q^{\Delta(d)-\Delta(f)-3}\epsilon^{fghd}e^\lambda_f \mathtt{e}\ , \tag{3.62}$$

which follows from eqs.(3.57) and (3.58). Neglecting the overall factor of $q^{-\Delta(a)-\Delta(b)-3}\mathtt{e}$ in (3.61) and multiplying it on the right by e^d_λ we finally get

$$q^{-\Delta(c)}\mathcal{T}^c_{bc}\ \delta^d_a + q^{-\Delta(c)}\mathcal{T}^c_{ca}\ \delta^d_b + q^{-\Delta(d)}\mathcal{T}^d_{ab} = 0\ . \tag{3.63}$$

It is easy to verify that these equations imply the vanishing of all the $\mathcal{T}^a_{bc}$ and thus of the torsion.

The Christoffel symbols $\Gamma^\sigma_{\mu\nu}$ are defined in the same way of the previous section, by demanding that the covariant derivative of the vierbeins vanishes,

$$\mathtt{D}_\mu e^b_\nu = 0\ . \tag{3.64}$$

The difference with the undeformed case is that we cannot switch the order of objects arbitrarily. To eliminate the spin–connection from (1.11) we now multiply on the left by $q^{\Delta(a)}\eta_{ab}e^a_\rho$, and proceed as in the undeformed case. We can then isolate $\Gamma^\sigma_{\mu\nu}$ according to

$$2q^{\Delta(a)}\eta_{ab}e^a_\rho e^b_\sigma\Gamma^\sigma_{\mu\nu} = q^{\Delta(a)}\eta_{ab}[e^a_\rho(\partial_\mu e^b_\nu + \partial_\nu e^b_\mu) + e^a_\nu(\partial_\mu e^b_\rho - \partial_\rho e^b_\mu) + e^a_\mu(\partial_\nu e^b_\rho - \partial_\rho e^b_\nu)] \tag{3.65}$$

or

$$2\mathtt{g}_{\rho\sigma}\Gamma^\sigma_{\mu\nu} = \partial_\mu\mathtt{g}_{\rho\nu} + \partial_\nu\mathtt{g}_{\rho\mu} - \partial_\rho\mathtt{g}_{\nu\mu}\ . \tag{3.66}$$

To solve this equation we need the inverse of the metric $\mathtt{g}^{\mu\nu}$. The expression

$$\mathtt{g}^{\mu\nu} = q^{\Delta(a)}\eta^{ab}e^\mu_a e^\nu_b\ , \tag{3.67}$$

does the job as it can be checked that

$$\mathtt{g}^{\mu\rho}\mathtt{g}_{\rho\nu} = \mathtt{g}_{\nu\rho}\mathtt{g}^{\rho\mu} = \delta^\mu_\nu\ . \tag{3.68}$$

Notice that unlike in the usual Einstein Cartan theory

$$\mathtt{g}^{\mu\nu}\eta_{ab}e^b_\nu = q^{\Delta(a)}e^\mu_a\ . \tag{3.69}$$

We are now able to solve eq.(3.66). Upon multiplying it by $\mathtt{g}^{\tau\rho}$ on each side, we get the usual expression for the Christoffel symbols in terms of the metric tensor and its inverse. It may be verified, using these expressions, that the Christoffel symbols commute with everything and

thus, even if written in terms of non-commuting quantities, they can be interpreted as being ordinary numbers.

The covariant derivative operator ∇_μ defined by the Christoffel symbols is compatible with the metric $\mathtt{g}_{\mu\nu}$, i.e. $\nabla_\mu \mathtt{g}_{\nu\rho} = 0$. This is clear because our Christoffel symbols have the standard expression in terms of the space-time metric $\mathtt{g}_{\mu\nu}$, but also follows from eq.(3.64)

$$\nabla_\mu \mathtt{g}_{\nu\rho} = \mathtt{D}_\mu \mathtt{g}_{\nu\rho} = \mathtt{D}_\mu (q^{\Delta(a)} \eta_{ab} e^a_\nu e^b_\rho) = 0 \ . \tag{3.70}$$

We now construct the Riemann tensor. It is defined as in the undeformed theory:

$$\mathtt{R}_{\mu\nu\rho}{}^\sigma v_\sigma = (\mathtt{D}_\mu \mathtt{D}_\nu - \mathtt{D}_\nu \mathtt{D}_\mu) v_\rho \ , \tag{3.71}$$

where v_μ is an ordinary co-vector. It follows from (3.64) that it has the standard expression in terms of the Christoffel symbols (and thus in terms of the space-time metric and its inverse) and therefore its components commute with everything. (This is also true for the Ricci tensor $\mathtt{R}_{\mu\nu} = \mathtt{R}_{\mu\sigma\nu}{}^\sigma$, of course, but not for $\mathtt{R}_{\mu\nu\rho\tau}$ as the lowering of the upper index of the Riemann tensor implies contraction with $\mathtt{g}_{\sigma\tau}$ which is not in the center of the algebra). The relation among the Riemann tensor and the curvature of the spin connection follows from eq. (3.64):

$$e^a_\sigma \mathtt{R}_{\mu\nu\rho}{}^\sigma v^\rho = e^a_\sigma (\mathtt{D}_\nu \mathtt{D}_\mu - \mathtt{D}_\mu \mathtt{D}_\nu) v^\sigma = (\mathcal{D}_\nu \mathcal{D}_\mu - \mathcal{D}_\mu \mathcal{D}_\nu) e^a_\sigma v^\sigma = -\mathcal{R}^{ac}_{\mu\nu} \eta_{bc} e^b_\sigma v^\sigma \ , \tag{3.72}$$

$\mathcal{R}^{ab}_{\mu\nu}$ being the space-time components of $\mathcal{R}^{ab}$. v^μ being an arbitrary ordinary vector, it follows from the above equation that:

$$\mathtt{R}_{\mu\nu\rho}{}^\tau = -\mathcal{R}^{ac}_{\mu\nu} \eta_{bc} e^b_\rho e^\tau_a \ . \tag{3.73}$$

Using this equation it can be checked directly that the components of the Riemann tensor commute with everything, as pointed out earlier. Our Riemann tensor has the usual symmetry properties:

$$\begin{aligned} \mathtt{R}_{\mu\nu\rho}{}^\sigma &= -\mathtt{R}_{\nu\mu\rho}{}^\sigma \ , \\ \mathtt{R}_{\mu\nu\rho\sigma} &= -\mathtt{R}_{\mu\nu\sigma\rho} \ , \\ \mathtt{R}_{[\mu\nu\rho]}{}^\sigma &= 0 \ . \end{aligned} \tag{3.74}$$

The first of these equations is obvious; the second can be proved starting from (3.73):

$$\begin{aligned} \mathtt{R}_{\mu\nu\rho\sigma} &= \mathtt{R}_{\mu\nu\rho}{}^\tau \mathtt{g}_{\tau\sigma} = -\mathcal{R}^{ab}_{\mu\nu} e_{\rho b} e^\tau_a \mathtt{g}_{\tau\sigma} = \\ &= -q^{\Delta(a)} \mathcal{R}^{ab}_{\mu\nu} e_{\rho b} e_{\sigma a} = -q^{\Delta(b)} \mathcal{R}^{ab}_{\mu\nu} e_{\sigma a} e_{\rho b} = -\mathtt{R}_{\mu\nu\sigma\rho} \ , \end{aligned} \tag{3.75}$$

where we have made use of (3.69). The third of eqs.(3.74) follows from the algebraic Bianchi identity and from (3.73):

$$0 = -\epsilon^{\lambda\mu\nu\rho} \mathcal{R}^{ac}_{\mu\nu} \eta_{bc} e^b_\rho = \epsilon^{\lambda\mu\nu\rho} \mathtt{R}_{\mu\nu\sigma}{}^\tau e^a_\tau e^\sigma_b e^b_\rho = 6 \, \epsilon^{\lambda\mu\nu\rho} \mathtt{R}_{[\mu\nu\rho]}{}^\tau e^a_\tau \ . \tag{3.76}$$

We now show that the action (2.42) becomes equal to the *undeformed* Einstein-Hilbert action, once the spin connection is eliminated using its equations of motion, namely the zero

torsion condition. As in the undeformed case, first we rewrite (2.42) in a form analogous to Palatini's action, and then show that the latter reduces to the *undeformed* Einstein-Hilbert action, once the spin-connection is eliminated from it. Consider thus the following deformation of the Palatini action:

$$\mathcal{S} = \frac{1}{2} \int_M d^4x \, q^{\Delta(a)-3} \mathtt{e} \, e_a^\mu e_b^\nu \mathcal{R}_{\mu\nu}^{ab} \; . \tag{3.77}$$

To see that it coincides with (2.42), we use the identity:

$$q^{\Delta(a)-\Delta(b)-6} \hat{\epsilon}^{abcd} e_a^\mu e_b^\nu \mathtt{e} = -\epsilon^{\mu\nu\lambda\sigma} e_\lambda^c e_\sigma^d \; . \tag{3.78}$$

The result (3.77) then follows after multiplying both sides of this equation on the left by $-1/8 \; q^{-2\Delta(f)-\Delta(g)-3} \hat{\epsilon}_{fgcd} \mathcal{R}_{\mu\nu}^{fg}$ and using the identity

$$\hat{\epsilon}_{fgcd} \hat{\epsilon}^{abcd} = -2q^6 \; (\delta_f^a \delta_g^b - q^{\Delta(f)-\Delta(g)} \delta_g^a \delta_f^b) \; ,$$

along with (3.56).

We now show that eq.(3.77) becomes in turn equal to the undeformed Einstein-Hilbert action upon eliminating the spin connection via its equation of motion. This amounts to expressing $\mathcal{R}_{\mu\nu}^{ab}$ in terms of the Riemann tensor by inverting eq.(3.73) and then plugging the result in eq.(3.77). We have:

$$\begin{aligned} & q^{\Delta(a)} e_a^\mu e_b^\nu \mathcal{R}_{\mu\nu}^{ab} = -q^{\Delta(a)} \mathtt{R}_{\mu\nu\rho}{}^{\tau} e_a^\mu e_b^\nu e_\tau^a e^{b\rho} = \\ & = -q^{\Delta(b)} \mathtt{R}_{\mu\nu\rho}{}^{\mu} e_b^\nu e^{b\rho} = \mathtt{R}_{\nu\mu\rho}{}^{\mu} \mathtt{g}^{\nu\rho} = \mathtt{R} \; , \end{aligned} \tag{3.79}$$

where we have made use of (3.59). Moreover we get, after a cumbersome calculation:

$$\mathtt{g} \equiv \det \| \, \mathtt{g}_{\mu\nu} \, \| = \frac{1}{4!} \epsilon^{\mu_1\mu_2\mu_3\mu_4} \epsilon^{\nu_1\nu_2\nu_3\nu_4} \mathtt{g}_{\mu_1\nu_1} \mathtt{g}_{\mu_2\nu_2} \mathtt{g}_{\mu_3\nu_3} \mathtt{g}_{\mu_4\nu_4} = q^{-6} \, \mathtt{e}^2 \; , \tag{3.80}$$

Putting together (3.79) and (3.80) we see that the q-Palatini action (3.77) becomes equal to:

$$\mathcal{S} = \frac{1}{2} \int_M d^4x \, \sqrt{-\mathtt{g}} \, \mathtt{R} \; , \tag{3.81}$$

which is the *undeformed* Einstein-Hilbert action. Since the components of $\mathtt{g}_{\mu\nu}$ and its inverse all commute among themselves, it is clear that the equations of motion of the metric theory will be equal to those of the undeformed Einstein's theory in vacuum. One can obtain the same result starting directly from eq.(2.45) and using (3.73).

4 Conclusions

From the results of the last section we may conclude that if we just consider the theory constructed in terms of the space-time metric $\mathtt{g}_{\mu\nu}$, ignoring the underlying gauge formulation, our theory is completely equivalent to Einstein's theory. No trace of the non–commutative structure existing in the gauge formulation of the theory can be found at the metric level.

Though the metric itself is non–commutative, as it doesn't commute with the connection components, all the physical objects constructed out of it, e. g. the Christoffel symbols together with the Riemann, Ricci and Einstein tensors, are c-number. Thus it appears that, at the level of *classical* General Relativity we can choose whatever representative of the one parameter family of q–gauge theories (not only the well known $q = 1$ theory) without changing the physics we are describing. That is, we have discovered a non–commutative structure in General Relativity which is hidden, even in the presence of matter, provided there are no sources of torsion. The possible physical implications of such an hidden structure need however further, careful investigation. We will report on these aspects in a forthcoming paper.

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